NOVA SCHOOL OF BUSINESS & ECONOMICS

Mathematics Written Exam

Date: May 21, 2018

Duração: 2 hours

Instructions:

- This exam has two groups: I and II.
- Group I has seven multiple choice questions and four possible answers to each one, of which only one is correct. In each question, circle the answer you think is correct, not showing the calculations made. If you circle more than one answer for the same question, the question will be considered unanswered. Each correct answer is worth 1 point, each incorrect answer is worth $-\frac{1}{3}$ points and each blank or invalid answer is worth 0 points. The minimum total number of points of this group is 0.
- Group II has three open – ended questions, the first with four parts, the second with five parts and the third with two parts. The grade of each question is written before its text. Show all the calculations and justify all the reasonings made. If you need to round up numbers in intermediate steps, use two decimal places. Answer each question in the correspondent space and use the front and back of each sheet.
- No questions will be answered during the exam. If you need to assume something while answering a question, state it, and be consistent with what you assumed in the steps that follow.
- Only use writing material and a calculator. Do not use cell phones or other material.
- Do not unstaple this exam.
- The last two pages of this exam have formulae and space for drafts, whose contents will not be graded.

Name:

Group I

1 Let Ω be the space result associated to a certain random experience and let A and B be two events $(A \subseteq \Omega$ and $B \subseteq \Omega)$. It is known that $P(A \cup B) = P(A) + P(B)$. Which of the following statements is necessarily true?

a) $P(A \cap B) > 0.$

b) $P(\bar{A} \cup B) = 1.$

c) $P(A \backslash B) = 0.$

d) $A \cup B = \Omega.$

2 Maria is at an ATM and is going to withdraw money using a card she just received. However, she lost the paper with the four-digit code, to which she only looked once. She remembers that the code had the digits 1, 2, 3 and 4, but does not remember their order. As she is in a hurry, she will make random attempts until she is able to withdraw money, or the card gets blocked in the machine. If she types a wrong code, she won't try it in the remaining attempts. The machine accepts three wrong codes, but, if there is a fourth wrong code, it blocks the card. What is the probability that Maria's card will be blocked by the machine?

a) $\frac{3}{4}.$

b) $\frac{23}{24}.$

c) $\frac{1}{4}.$

d) $\frac{5}{6}.$

3 Let f be a real function of a real variable, with domain $\mathbb{R}$, and f' and f'' its first and second derivatives, respectively, both with domain $\mathbb{R}$. Knowing that $\lim_{x \to -\infty} \left(f'(x)\right) = 3$ and that f'' is positive, which of the following statements is necessarily true?

a) $\lim_{x \to -\infty} \left(f(x)\right) = 3$

b) f is strictly increasing.

c) f is positive.

d) f has an inflection point.

4 Let f be the real function of a real variable, with domain $\mathbb{R}$, defined by $f(x) = x - 2$, and g a real function of a real variable, with domain $\mathbb{R}$. Knowing that g is continuous and even, that $g(-1) < 0$ and that $g(3) > 0$, in which of the following intervals is there, necessarily, a zero of function $f \times g$?

a) $[-3; 1]$.

b) $[-\infty; -1]$.

c) $[-1; 3]$.

d) $[1; +\infty]$.

5 Let f be a real function of a real variable, with domain $\mathbb{R}$, and g the real function of a real variable defined by $g(x) = |3 + 2f(x+1)|$. Knowing that $\lim_{x \to +\infty} \left(f(x)\right) = -5$, what is the value of $\lim_{x \to +\infty} \left(g(x)\right)$?

a) -5.

b) 5.

c) 13.

d) 7.

6 A firm has to choose how many watches to sell, in order to get the highest possible profit with the watches sale. The price the consumers are willing to pay for each watch depends on the total quantity of sold watches, and is given by $p(x) = 10 - x$, where x is the quantity of sold watches, and p is in €. Each watch has a production cost of 2€. If the firm produces the quantity of watches which maximizes the profit, which profit will it get?

a) 0€.

b) 25€.

c) 16€.

d) 4€.

7 Let z be a complex number such that its conjugate and its symmetric are the same. Which of the following statements is necessarily true?

a) $z \in \mathbb{R}$.

b) $\mathrm{Re}(z) = 0$.

c) $\arg(z) = \frac{\pi}{4}$.

d) $|z| = 1$.

Group II

1 **(5)** A bag has four balls in which it can be read, respectively, 1, 2, 3 and 4. Luís is going to randomly draw a ball from the bag and toss a balanced coin the number of times read in the ball drawn. In each coin toss, the result may be heads or tails.

a) **(1)** Let X be the random variable "Number of heads Luís gets, if the ball with number 2 is drawn". Find the average of X.

b) **(1,5)** What is the probability that Luís gets 3 heads in the coin tosses he does? Show the result as a decimal number.

c) **(1)** Knowing Luís got 3 heads in the coin tosses he did, what is the probability the ball with number 4 was drawn? Show the result as a simplified fraction.

d) **(1,5)** After the coin tosses, Luís put in the bag the drawn ball and two other objects he had in his pocket, none of them a ball. Now, he is going to randomly draw an object from the bag. Consider events A: "The object drawn from the bag is yellow" and B: "The object drawn from the bag is a ball". Knowing that $P(\overline{A \cup B}) = P(A \cap B)$ and that $P(B \backslash A) = \frac{1}{3}$, find the number of yellow balls.

Answer Question 1

2 **(6)** Let f be the real function of a real variable, with domain D_f, defined by $f(x) = \ln(x^3 + 3x^2)$ and g be the real function of a real variable, with domain D_g, defined by $g(x) = \frac{3x-3+\sin(x-1)}{2x-2}$.

a) **(1)** If the domains of f and g are the $\mathbb{R}$ subsets which have all the points which have images in f and g, respectively, find D_f and D_g.

b) **(1,5)** Study the monotonicity and existence of relative extrema of f.

c) **(1,5)** Study the concavity of the graph and the existence of inflection points of f.

d) **(2)** Let h be the real function of a real variable, with domain $\mathbb{R}^+$, defined by:

$$h(x) = \begin{cases} \dfrac{f(x)}{\ln(2)} & \text{if} \quad x \leq 1 \\ g(x) & \text{if} \quad x > 1 \end{cases}$$

(i) **(1)** Check if h is continuous at $x = 1$.

(ii) **(1)** Study the existence of vertical asymptotes of h.

Answer Question 2

3 **(2)** In $\mathbb{C}$, the set of complex numbers, consider $w = 2cis\left(\frac{5}{4}\pi\right)$.

a) **(1)** Find the area of the zone of the complex plane defined by:

$$\arg(\overline{w}) \leq \arg(z) \leq \arg(w) \wedge \mathrm{Re}(z) \geq \mathrm{Re}(w) \wedge \mathrm{Im}(z) \geq 0$$

b) **(1)** Let s be a complex number with the same argument as w. Show that the polygon whose vertices are the representations of the fourth roots of s^4 in the complex plane is a square whose sides are parallels to the axes.

Answer Question 3

Formulae

Special Limits

$$
\begin{array}{rcll}
\lim\left(\left(1+\frac{1}{n}\right)^n\right) & = & 1 & (n \in \mathbb{N}) \\
\lim\limits_{a\to 0}\left(\frac{\sin(a)}{a}\right) & = & 1 & \\
\lim\limits_{a\to 0}\left(\frac{e^a-1}{a}\right) & = & 1 & \\
\lim\limits_{a\to 0}\left(\frac{\ln(a+1)}{a}\right) & = & 1 & \\
\lim\limits_{a\to +\infty}\left(\frac{\ln(a)}{a}\right) & = & 0 & \\
\lim\limits_{a\to +\infty}\left(\frac{b^a}{a^p}\right) & = & +\infty & (b > 1, p \in \mathbb{R})
\end{array}
$$

Derivation Rules

$$
\begin{array}{rcll}
(a+b)' & = & a' + b' & \\
(ab)' & = & a'b + ab' & \\
\left(\frac{a}{b}\right)' & = & \dfrac{a'b - ab'}{b^2} & \\
(a^p)' & = & pa^{p-1}a' & (p \in \mathbb{R}) \\
(p^a)' & = & \ln(p)\, p^a a' & (p \in \mathbb{R}^+ \setminus \{1\}) \\
\left(\log_p(a)\right)' & = & \dfrac{a'}{\ln(p)\, a} & (p \in \mathbb{R}^+ \setminus \{1\}) \\
(\sin(a))' & = & a' \cos(a) & \\
(\cos(a))' & = & -a' \sin(a) & (b > 1, p \in \mathbb{R})
\end{array}
$$

Complex Numbers

$$
\begin{array}{rcll}
\left(\rho \operatorname{cis}(\theta)\right)^n & = & \rho^n \operatorname{cis}(n\theta) & (n \in \mathbb{N}) \\
\sqrt[n]{\rho \operatorname{cis}(\theta)} & = & \sqrt[n]{\rho} \operatorname{cis}\left(\dfrac{\theta + 2k\pi}{n}\right) & (n \in \mathbb{N}, k \in \{0, \dots, n-1\})
\end{array}
$$

Drafts